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A universal model for cosymplectic manifolds ★

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Abstract

We show that "all" cosymplectic manifolds can be obtained by reduction from a universal cosymplectic manifold $\mathbb{R} \times T^*(\mathbb{R}^N \times \mathbf{T}^k)$. We also prove a corresponding equivariant version.

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1. Introduction

It is by now a generally accepted statement that symplectic manifolds are the natural geometric setting for (time independent) classical mechanics. In [GT1] it was shown that the local model of a symplectic manifold, i.e., $\mathbb{R}^{2N}$ with its canonical symplectic form $d\theta_{\mathbb{R}^N}$, also serves as a universal model in the sense that "all" symplectic manifolds can be obtained from an $\mathbb{R}^{2N}$ by reduction (in a precise sense). On the other hand, cosymplectic manifolds are the natural geometric setting for time-dependent mechanics [LR,A], or, said differently, cosymplectic manifolds are a natural odd-dimensional counterpart to symplectic manifolds. Since the local model for a cosymplectic manifold is $\mathbb{R}^{2n+1}$ with the 2-form $d\theta_{\mathbb{R}^n}$ and the 1-form ds, it is quite natural to ask whether this is at the same time a universal model. Unlike the case of symplectic manifolds, the answer is negative: a universal model for cosymplectic manifolds is slightly more complicated.

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Let (M, Ω, η) be a cosymplectic manifold with Reeb vector field $\mathcal{R}$ and let C be a submanifold of M. Suppose furthermore that $\mathcal{R}$ is tangent to C, that the characteristic distribution $\mathcal{F} \equiv \ker \Omega|_C \cap \ker \eta|_C$ has constant rank, and that the canonical projection $\pi : C \to C/\mathcal{F} \equiv M_r$ is a fibration. With these hypotheses M_r inherits a canonical cosymplectic structure, and we will say that M_r is the reduction of M by C. The aim of this paper is to show that there exists a universal cosymplectic manifold in the sense that "all" cosymplectic manifolds may be obtained from it by reduction. More precisely, our main theorem is:

Theorem 3.1. *Let (M, Ω, η) be a cosymplectic manifold of finite type. Then there exist integers N and k and real numbers $\mu_1, \dots, \mu_k$ that are independent over $\mathbb{Q}$ such that M is the reduction of the cosymplectic manifold (M_u, Ω_u, η_u) by some submanifold $C \subset M_u$, where*

$$M_u = \mathbb{R} \times T^*(\mathbf{T}^k \times \mathbb{R}^N), \qquad \Omega_u = \mathrm{d}\theta_{\mathbf{T}^k \times \mathbb{R}^N}, \qquad \eta_u = \mathrm{d}s + \sum_{i=1}^{k} \mu_i \, \mathrm{d}\varphi_i,$$

with φ_i the angle coordinates on the torus $\mathbf{T}^k$, s the canonical coordinate in $\mathbb{R}$, and $k =$ $\mathrm{rank}(\eta)$.

Now suppose that the cosymplectic manifold M_r is obtained by reduction from the cosymplectic manifold M by the submanifold $C \subset M$. If a Lie group G acts cosymplectically on (M, Ω, η) and if C is invariant under G, the action of G on M induces a cosymplectic action of G on M_r. In such a case we say that M_r is obtained as the equivariant reduction of M by C. This happens, for instance, in the context of cosymplectic reduction of a cosymplectic manifold with symmetry (see [A,CLL,LS]). Theorem 3.1 can be improved to incorporate (compact) group actions as given in:

Theorem 4.2. *Let G be a Lie group acting cosymplectically on a connected cosymplectic manifold (M, Ω, η), where M is of finite type. If the closed 2-form Ω admits a moment map, then (M, Ω, η, G) can be obtained as the equivariant reduction of some cosymplectic manifold (M_u, Ω_u, η_u) as in Theorem 3.1 equipped with a (cosymplectic) action of G, which is trivial on $\mathbb{R}$ and which is the symplectic lift of an action on $\mathbf{T}^k \times \mathbb{R}^N$, the latter action decomposing as a representation of G on $\mathbf{T}^k$ and an orthogonal representation of G on $\mathbb{R}^N$.*

2. Reduction of a cosymplectic manifold

Before we give the formal definition of cosymplectic reduction used in this paper, we briefly recall the definition and some important properties of cosymplectic manifolds. A *cosymplectic manifold* is a triple (M, Ω, η) consisting of a smooth $(2n+1)$-dimensional manifold M, endowed with a closed 2-form Ω and a closed 1-form η such that $\Omega^n \wedge \eta$ is nowhere zero (see [LM,LR]). If we consider the vector bundle morphism

$$b : TM \to T^*M, \qquad X \in T_xM \mapsto b(X) = \iota(X)\Omega_x + (\iota(X)\eta_x)\eta_x,$$

then the condition that $\Omega^n \wedge \eta$ is nowhere zero is equivalent to the condition that b is a vector bundle isomorphism. One denotes by $\mathcal{R} = b^{-1}(\eta)$ the *Reeb vector field*, which means that it is defined by

$$\iota(\mathcal{R})\Omega = 0 \quad \text{and} \quad \iota(\mathcal{R})\eta = 1.$$

Around any point $x \in M$ there exist local canonical coordinates (q^a, p_a, u), $a = 1, \dots, n$, called *Darboux coordinates*, such that

$$\Omega = \sum_{a=1}^{n} \mathrm{d}q^a \wedge \mathrm{d}p_a, \qquad \eta = \mathrm{d}u.$$

In such coordinates the Reeb vector field is given as $\mathcal{R} = \partial_u$.

Two cosymplectic manifolds (M_1, Ω_1, η_1) and (M_2, Ω_2, η_2) are said to be isomorphic if there exists a diffeomorphism $\Phi : M_1 \to M_2$ such that

$$\Phi^*\Omega_2 = \Omega_1, \qquad \Phi^*\eta_2 = \eta_1.$$

Now let (M, Ω, η) be a cosymplectic manifold and let C be a submanifold. By $\Omega|_C$ and $\eta|_C$ we denote the restrictions of Ω and η to C, respectively. We furthermore assume that the following three conditions are satisfied:
- $\mathcal{R}$ is tangent to C;
- the characteristic distribution $\mathcal{F} = \ker \Omega|_C \cap \ker \eta|_C$ has constant rank on C (hence it is a foliation on C);
- the space of leaves $M_r = C/\mathcal{F}$ has a structure of manifold and the canonical projection $\pi : C \to M_r$ is a fibration.

With these hypotheses, it is not hard to show that there exist unique closed forms Ω_r and η_r on M_r such that:
- $\pi^*\Omega_r = \Omega|_C$ and $\pi^*\eta_r = \eta|_C$;
- (M_r, Ω_r, η_r) is a cosymplectic manifold;
- $\pi_*(\mathcal{R}|_C) = \mathcal{R}_r$, where $\mathcal{R}|_C$ is the restriction of $\mathcal{R}$ to C and $\mathcal{R}_r$ the Reeb vector field of the cosymplectic manifold M_r.

In these circumstances we will say that M_r is the *reduction of M by C*.

Remark 2.1. In the tangent space of a cosymplectic manifold M we can define a cosymplectic orthogonal by

$$E \subset T_xM \quad \Longrightarrow \quad E^\perp = \{\, X \in T_xM \mid \eta(X) = 0\ ,\ (\iota(X)\Omega)|_E = 0 \,\}.$$

With this we say that a submanifold $C \subset M$ is *coisotropic* if: (i) $\mathcal{R}$ is tangent to C; and (ii) $TC^\perp \subset TC$. Since the dimension of $E^\perp$ only depends on the dimension of E, it follows easily that if C is coisotropic, then $\mathcal{F} = TC \cap TC^\perp$ has constant dimension.

The notion of a coisotropic submanifold C of a cosymplectic manifold M can be related to the usual notion of a coisotropic submanifold of a symplectic manifold in the following way. If (M, Ω, η) is cosymplectic, then the manifold $M \times \mathbb{R}$ equipped with the closed 2-form $\omega = \Omega + \eta \wedge \mathrm{d}t$ is a symplectic manifold (here t is the coordinate on the extra $\mathbb{R}$).

One then can prove that $C \subset M$ is coisotropic in the cosymplectic sense if and only if $T(C \times \mathbb{R})^{\perp} \subset TC \times \{0\}$, where one should use the symplectic orthogonal to compute $T(C \times \mathbb{R})^{\perp}$. It follows that $C \times \mathbb{R}$ is a special kind of coisotropic submanifold of $M \times \mathbb{R}$ (in the symplectic sense).

Having defined what we mean by (cosymplectic) reduction, we now proceed to collect some properties of this kind of reduction.

Proposition 2.2. *Suppose that a cosymplectic manifold M_2 is the reduction of a cosymplectic manifold M_1 by a submanifold $C_1 \subset M_1$ and that M_1 is the reduction of a cosymplectic manifold M_0 by the submanifold $C_0 \subset M_0$. Then M_2 is the reduction of M_0 by the submanifold $C_2 = \pi_0^{-1}(C_1)$, where $\pi_i : C_i \to M_{i+1}$ is the canonical projection, $i = 0, 1$.*

Proof. A direct computation. □

Definition 2.3. For any 1-form η on a manifold M we define $\operatorname{rank}(\eta)$ by

$$\operatorname{rank}(\eta) = \dim_{\mathbb{Q}}(\mathbb{Q} \cdot \operatorname{Periods}(\eta)),$$

where $\operatorname{Periods}(\eta) = \{\int_z \eta \mid z \text{ a 1-chain in } M\} \subset \mathbb{R}$. Note that for manifolds of finite type (meaning that $H_q(M)$ is finitely generated for all $q \in \mathbb{N}$, a condition verified by all reasonable manifolds) the $\mathbb{Q}$-dimension of $\mathbb{Q} \cdot \operatorname{Periods}(\eta)$ is always finite.

Proposition 2.4. *Let M_r be the reduction of M by C, and let* $\operatorname{rank}(\eta) = k$. *Then* $\operatorname{rank}(\eta_r) \leq k$.

Proof. We have $\operatorname{rank}(\eta|_C) \leq \operatorname{rank}(\eta)$ because a chain in C is necessarily a chain in M. If $\pi : C \to M_r$ is the canonical projection, then a chain in C projects onto a chain in M_r and any chain in M_r is the projection of a chain in C (because the fibres of π are connected). Thus $\operatorname{rank}(\eta|_C) = \operatorname{rank}(\eta_r)$. □

Corollary 2.5. *If η is exact then η_r is also exact.*

Proof. η is exact if and only if $\operatorname{Periods}(\eta) = \{0\}$, i.e., if and only if $\operatorname{rank}(\eta) = 0$. □

Proposition 2.6. *Let M_r be the reduction of M by C. If the Reeb vector field $\mathcal{R}$ is complete and if C is closed in M, then $\mathcal{R}_r$ is complete.*

Proof. If C is closed, the restriction of $\mathcal{R}$ to C is a complete vector field on C which projects onto the reduced Reeb vector field $\mathcal{R}_r$. Hence $\mathcal{R}_r$ is complete. □

Remark 2.7. The local model for cosymplectic manifolds is given by the Darboux coordinates as $\mathbb{R}^{2n+1}$. In this local model the 1-form is exact, and the Reeb vector field is complete. Since for a general cosymplectic manifold the 1-form need not be exact, Corollary 2.5 shows that the local model cannot be a general universal model. Moreover, since the Reeb vector field of a general cosymplectic manifold need not be complete, it follows from Proposition 2.6 that we cannot always take the submanifold C to be closed. This is in sharp contrast with the symplectic case, in which the local model $\mathbb{R}^{2n}$ is at the same time the universal model, and in which reduction can always be done by a closed submanifold.

3. The main theorem

Theorem 3.1. *Let (M, Ω, η) be a cosymplectic manifold of finite type. Then there exist integers N and k and real numbers $\mu_1, \dots, \mu_k$ that are independent over $\mathbb{Q}$ such that M is the reduction of the cosymplectic manifold (M_u, Ω_u, η_u) by some submanifold $C \subset M_u$, where*

$$M_u = \mathbb{R} \times T^*(\mathbf{T}^k \times \mathbb{R}^N), \qquad \Omega_u = \mathrm{d}\theta_{\mathbf{T}^k \times \mathbb{R}^N}, \qquad \eta_u = \mathrm{d}s + \sum_{i=1}^{k} \mu_i \, \mathrm{d}\varphi_i,$$

with φ_i the angle coordinates on the torus $\mathbf{T}^k$, s the canonical coordinate in $\mathbb{R}$, and $k =$ $\operatorname{rank}(\eta)$.

Proof. The proof will proceed in four steps. In the first step we "concentrate" the 1-form on a torus; in the second step we transform the 2-form into a part of a symplectic form; in the third step we change coordinates to get a better view; finally in the fourth step we obtain the model as described in the theorem.

Step 1: The condition that M is of finite type implies in particular that the 1-form η can be written as a finite linear combination of integral 1-forms:

$$\eta = \eta_0 + \sum_{i=1}^{k} \mu_i \, \eta_i,$$

where η_0 is exact and where the η_i are non-zero integral classes (note that if η is exact, $k = 0$, and if $k > 0$ then we can absorb η_0 in one of the η_i; the given presentation allows us to present both cases at the same time). If we take the number of integral classes k-minimal, the coefficients μ_i must be independent over $\mathbb{Q}$. In that case it follows immediately that $\operatorname{rank}(\eta) = k$.

Now recall that S^1 is an Eilenberg–MacLane space ($K(\mathbb{Z}, 1) = S^1$). Thus, from the Eilenberg Classification theorem [W], the set of homotopy classes of mappings from M into $S^1 \equiv \mathbf{T}$ is in one-to-one correspondence with the group $H^1(M, \mathbb{Z})$. Since $[\eta_i] \in H^1(M, \mathbb{Z})$, there exist smooth maps $f_i : M \to \mathbf{T}$ such that $f_i^*(\mathrm{d}\varphi) = \eta_i$.

With this preparation we define (M_1, Ω_1, η_1) by:

$$M_1 = M \times T^*\mathbf{T}^k \cong (M \times \mathbb{R}^k) \times \mathbf{T}^k,$$

$$\Omega_1 = \Omega + \sum_{i=1}^{k} \mu_i \, \mathrm{d}t_i \wedge (\mathrm{d}\varphi_i - \eta_i), \qquad \eta_1 = \eta_0 + \left(\sum_{i=1}^{k} \mu_i \, \mathrm{d}\varphi_i \right),$$

where (φ_i) are the usual coordinates (modulo 2π) in the torus $\mathbf{T}^k$ and t_i the coordinates in the corresponding fibres $\mathbb{R}^k$. It is easy to show that $d\Omega_1 = 0$, that $d\eta_1 = 0$, and that

$$\iota(X)\Omega_1 + (\iota(X)\eta_1) \cdot \eta_1 = 0 \iff X = 0.$$

We conclude that (M_1, Ω_1, η_1) is a cosymplectic manifold. Its Reeb vector field is

$$\mathcal{R}_1 = \mathcal{R} + \sum_{i=1}^{k} (\iota(\mathcal{R})\eta_i) \cdot \frac{\partial}{\partial \varphi_i},$$

where $\mathcal{R}$ denotes the Reeb vector field of (M, Ω, η).

We now define the submanifold $C_1 = \{(x, t_i, f_i(x))\} \subset M_1$, which is the graph of the smooth map $F : M \times \mathbb{R}^k \to \mathbf{T}^k$ defined as $F(x, t_i) = (f_i(x))$. Hence C_1 is diffeomorphic to $M \times \mathbb{R}^k$. A direct computation shows that

$$\mathcal{F}_1 \equiv \ker \Omega_1|_{C_1} \cap \ker \eta_1|_{C_1} = \langle \partial/\partial t_i \rangle.$$

It follows that (M_r, Ω_r, η_r), the cosymplectic reduction of M_1 by C_1, is isomorphic to (M, Ω, η).

Step 2: We define

$$M_2 = \mathbb{R} \times T^*M_1, \qquad \Omega_2 = d\theta_{M_1} + \Omega_1, \qquad \eta_2 = ds + \eta_1.$$

A direct computation shows that (M_2, Ω_2, η_2) is a cosymplectic manifold whose Reeb vector field is $\mathcal{R}_2 = \partial_s$, where s denotes the canonical coordinate on $\mathbb{R}$. We denote by D_1 the domain of the flow of $\mathcal{R}_1$, which is an open subset of $\mathbb{R} \times M_1$. We then define the submanifold C_2 of M_2 by

$$C_2 = \{(s, x, 0_x) \in \mathbb{R} \times T^*M_1 \mid (s, x) \in D_1\}.$$

Roughly speaking C_2 is the zero section of T^*M_1 above D_1, and as such C_2 is diffeomorphic to D_1. Using this identification we obtain

$$\mathcal{F}_2 \equiv \ker \Omega_2|_{C_2} \cap \ker \eta_2|_{C_2} = \langle \mathcal{R}_1 - \partial/\partial s \rangle.$$

If the flow of $\mathcal{R}_1$ is given by $x \mapsto x(t)$, the flow of $\mathcal{R}_1 - \partial_s$ is given by $(s, x) \mapsto (s - t, x(t))$. It follows that each leaf of $\mathcal{F}$ has a unique point with $s = 0$. Hence the quotient $C_2/\mathcal{F}_2$ may be identified with $M_1 \cong \{0\} \times M_1 \subset D_1 \cong C_2$. An elementary computation then finishes the proof that (M_1, Ω_1, η_1) is the cosymplectic reduction of M_2 by C_2.

Step 3: Using the notation introduced in steps 1 and 2, we can define a 1-form α on M_1 by

$$\alpha = \sum_{i=1}^{k} \mu_i t_i (d\varphi_i - \eta_i).$$

Moreover, since η_0 is exact, there exists a smooth function $f_0 : M_1 \to \mathbb{R}$ such that $\eta_0 = d f_0$. With these ingredients, we can write the cosymplectic structure on M_2 as:

$$\Omega_2 = d\theta_{M_1} + d\alpha + \Omega, \qquad \eta_2 = ds + d f_0 + \sum_{i=1}^{k} \mu_i \, d\varphi_i.$$

We now introduce the diffeomorphism $g : M_2 \to M_2$ defined by

$$g(s, \beta_x) = (s - f_0(x), \beta_x - \alpha_x),$$

where $\beta_x \in T_x^* M_1$ denotes an arbitrary point in the cotangent bundle of M_1. In other words, g is a translation in $\mathbb{R}$ over $f_0(x)$ and a translation in the fibres of T^*M_1 over the 1-form α. It is not hard to show that the pull backs $\Omega_3 = g^*\Omega_2$ and $\eta_3 = g^*\eta_2$ are given by:

$$\Omega_3 = \mathrm{d}\theta_{M_1} + \Omega, \qquad \eta_3 = \mathrm{d}s + \sum_{i=1}^{k} \mu_i \, \mathrm{d}\varphi_i .$$

It follows that the cosymplectic manifold (M_2, Ω_2, η_2) is isomorphic (via g) to the cosymplectic manifold $(M_3 = M_2, \Omega_3, \eta_3)$.

Step 4: Reordering the different factors in the definition of (M_3, Ω_3, η_3), we can write these as:

$$\begin{aligned} M_3 &= \mathbb{R} \times T^*\mathbf{T}^k \times T^*(M \times \mathbb{R}^k), \\ \Omega_3 &= \mathrm{d}\theta_{\mathbf{T}^k} + \mathrm{d}\theta_{M\times\mathbb{R}^k} + \Omega, \\ \eta_3 &= \mathrm{d}s + \sum_{i=1}^{k} \mu_i \, \mathrm{d}\varphi_i . \end{aligned}$$

We now recall that any symplectic manifold of finite type can be obtained by reduction (in the symplectic sense) from a universal model $(T^*\mathbb{R}^N, \mathrm{d}\theta_{\mathbb{R}^N})$ (see [GT1]). We apply this to the manifold $T^*(M \times \mathbb{R}^k)$ with the symplectic form $\omega = \mathrm{d}\theta_{M\times\mathbb{R}^k} + \Omega$. There thus exists an $N \in \mathbb{N}$ and a submanifold C of $T^*\mathbb{R}^N$ such that the quotient $C/\ker\omega|_C$ is isomorphic to the symplectic manifold $T^*(M \times \mathbb{R}^k)$, where $\omega|_C$ denotes (as usual) the restriction of the canonical symplectic form $\mathrm{d}\theta_{\mathbb{R}^N}$ on $T^*\mathbb{R}^N$ to C.

With this in mind we define $M_u = \mathbb{R} \times T^*\mathbf{T}^k \times T^*\mathbb{R}^N$, $\Omega_u = \mathrm{d}\theta_{\mathbf{T}^k\times\mathbb{R}^N}$, and $\eta_u = \mathrm{d}s + \sum_{i=1}^{k} \mu_i \, \mathrm{d}\varphi_i$. It follows easily that the cosymplectic manifold (M_3, Ω_3, η_3) is the reduction of M_u by the submanifold $C_u = \mathbb{R} \times T^*\mathbf{T}^k \times C$. We complete the proof of the theorem by concatenating the four steps; this is done by applying the chain rule proved in Proposition 2.2. □

Remark 3.2. The proof that all symplectic manifolds can be obtained as a reduction from some canonical $\mathbb{R}^{2N}$ decomposes into three steps. Comparing these steps with the four steps in the proof of Theorem 3.1, one can see that the first step in the symplectic case corresponds to the second step above, and that the second and third steps in the symplectic case correspond to the fourth step above.

Actually, this comparison allows us to be more precise about the fourth step above: (i) the submanifold C can be chosen to be coisotropic (in the symplectic sense); and (ii) the reduction from $\mathbb{R}^{2N}$ to $T^*(M \times \mathbb{R}^k)$ can be realized as a Marsden–Weinstein reduction by means of a connected abelian group.

4. The equivariant case

Let (M, Ω, η) be a cosymplectic manifold and let G be a Lie group acting smoothly on M. We will say that G acts cosymplectically if the action preserves both Ω and η. Next, suppose that the cosymplectic manifold (M_r, Ω_r, η_r) is obtained as the reduction of M by $C \subset M$. If the Lie group G acts cosymplectically on M and if C is invariant under this action, then there is a naturally induced action of G on M_r and, moreover, this induced action is cosymplectic. In such a case we will say that $(M_r, \Omega_r, \eta_r, G)$ is the equivariant reduction of (M, Ω, η, G) by C.

Definition 4.1. Let a Lie group G act smoothly on a manifold M, and let Ω be a closed G-invariant 2-form on M. If $\mathfrak{g}$ is the Lie algebra of G, then for any $\xi \in \mathfrak{g}$ we denote by ξ_M the associated fundamental vector field on M. Since Ω is G-invariant, the 1-forms $\iota(\xi_M)\Omega$ are closed. We will say that *the G-action admits a moment map* if all the closed 1-forms $\iota(\xi_M)\Omega$ are exact. This generalizes the usual notion of a moment map for symplectic manifolds.

Theorem 4.2. *Let G be a compact connected Lie group acting cosymplectically on a connected cosymplectic manifold (M, Ω, η), where M is of finite type. If the closed 2-form Ω admits a moment map, then (M, Ω, η, G) can be obtained as the equivariant reduction of some cosymplectic manifold (M_u, Ω_u, η_u) as in Theorem 3.1 equipped with a (cosymplectic) action of G, which is trivial on $\mathbb{R}$ and which is the symplectic lift of an action on $\mathbf{T}^k \times \mathbb{R}^N$, the latter action decomposing as a representation of G on $\mathbf{T}^k$ and an orthogonal representation of G on $\mathbb{R}^N$.*

Proof. We follow the steps of the proof of Theorem 3.1. By Lemma 4.3 we may assume that the integral 1-forms η_i are G-invariant. Applying Lemma 4.4 we find an action of G on $\mathbf{T}^k$ (induced by a representation $\rho : G \to \mathbf{T}^k$) such that the maps f_i are G-equivariant. We then equip $M_1 = M \times \mathbb{R}^k \times \mathbf{T}^k$ with a G-action by using the given action on M, the trivial action on $\mathbb{R}^k$ and the just constructed action on $\mathbf{T}^k$. It then follows easily that G acts cosymplectically on (M_1, Ω_1, η_1), that C is G-invariant, and that M is the equivariant reduction of M_1.

For the next step we define a G-action on $M_2 = \mathbb{R} \times T^*M_1$ by taking the trivial action on $\mathbb{R}$ and the canonical lift (from M_1) to the cotangent bundle T^*M_1. Since such a lifted action always preserves the canonical symplectic form $d\theta_{M_1}$, it follows that G acts cosymplectically on (M_2, Ω_2, η_2). Since the Reeb vector field on M_1 is G-invariant (it is defined in terms of Ω_1 and η_1), it follows that the domain D is G-invariant, and thus C_2 is G-invariant. It follows easily that M_1 is the equivariant reduction of M_2.

For step three it suffices to remark that α is G-invariant and that we may assume that f_0 is G-invariant (by an averaging argument). It follows that the isomorphism g is G-equivariant.

Finally, for step four, we note that the condition that Ω admits a moment map implies that we can apply the result of [GT2], i.e., there exists an orthogonal action of G on $\mathbb{R}^N$ such that the reduction (in the symplectic sense) from $\mathbb{R}^{2N}$ to $T^*(M \times \mathbb{R}^k)$ is an equivariant reduction. We thus define a G-action on $M_u = \mathbb{R} \times T^*\mathbf{T}^k \times T^*\mathbb{R}^N$ as being trivial on $\mathbb{R}$, the lifted action of G on $\mathbf{T}^k$ to the cotangent bundle $T^*\mathbf{T}^k$ (which is trivial on the fibres!), and the lifted orthogonal action of G on $T^*\mathbb{R}^N$. It follows immediately that M_3 is the equivariant reduction of M_u.

The proof is complete when we note that the chain rule of Proposition 2.2 is also valid in the equivariant setting. □

Lemma 4.3. *Let a compact and connected Lie group G act on a manifold M, and let the k-form η represent an integral cohomology class on M. Then the average $\bar{\eta} = \int_G g^*\eta \, \mathrm{d}g$ represents the same integral class, provided $\mathrm{d}g$ is the invariant Haar measure of total volume 1.*

Proof. For any k-cycle z we have $\int_z g^*\eta = \int_{g(z)} \eta = n(g) \in \mathbb{Z}$. Since G is connected and $\mathbb{Z}$ discrete, $n(g)$ must be constant. It follows that for any k-cycle z we have $\int_z \bar{\eta} = \int_z \eta$, and thus, by De Rham duality, $\bar{\eta}$ and η represent the same cohomology class. □

Lemma 4.4. *Let G be a Lie group acting smoothly on a connected manifold M, and let $f : M \to \mathbf{T}$ be a smooth map. If the 1-form $\eta = f^*\mathrm{d}\varphi$ is G-invariant, then there exists a representation ρ of G on $\mathbf{T}$ such that $f(gm) = \rho(g) \cdot f(m)$, i.e., f is equivariant if we equip $\mathbf{T}$ with the G-action defined by $g(z) = \rho(g) \cdot z$.*

Proof. Denoting as before the Lie algebra of G by $\mathfrak{g}$, G-invariance of η implies that $0 = \mathcal{L}(\xi_M)\eta = \mathrm{d}(\iota(\xi_M)\eta)$ for all $\xi \in \mathfrak{g}$. Since M is connected, this implies that $\iota(\xi_M)\eta$ must be constant, i.e., there exists a map $r : \mathfrak{g} \to \mathbb{R}$ such that

$$\forall \xi \in \mathfrak{g} : \iota(\xi_M)\eta = r(\xi).$$

Interpreting $\mathbb{R}$ as the (abelian) Lie algebra of the Lie group $\mathbf{T}$, it is not hard to show that the map r is a Lie algebra homomorphism. Thus there exists a (unique) Lie group morphism $\tilde{\rho} : \tilde{G} \to \mathbf{T}$ whose derivative at the identity equals r (where $\tilde{G}$ denotes the simply connected cover of G).

We now introduce two maps $\chi, \psi : \tilde{G} \times M \to M$ defined as:

$$\chi(\tilde{g}, m) = f(\pi(\tilde{g})m), \qquad \psi(\tilde{g}, m) = \tilde{\rho}(\tilde{g}) \cdot f(m),$$

where $\pi : \tilde{G} \to G$ denotes the covering map. Since for all $m \in M$ the maps χ and ψ coincide at $\tilde{g} = \mathrm{id}$ and have the same derivative at an arbitrary point $\tilde{g}$, we deduce that $\chi = \psi$. This means that we have the equality

$$f(\pi(\tilde{g})m) = \tilde{\rho}(\tilde{g}) \cdot f_i(m) \quad \forall \tilde{g} \in \tilde{G}, \quad \forall m \in M.$$

From this and the fact that $\mathbf{T}$ is a group, it follows immediately that there exists a (unique) representation $\rho : G \to \mathbf{T}$ such that $\tilde{\rho} = \rho \circ \pi$. This is the sought for ρ. □

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